Exercice (15):

- Methodology: 1°) Multiply the quasigeostrophic vorticity equation by x and integrate between $-\infty$ and $+\infty$
 2°) Look at each term in the equation obtain in 1°) and use the Gauss Theorem ("Divergence theorem") and the property of $aJ(b,c)$ to do some simplification.
 3°) Collect all the term and find (15.4)
 4°) Using the previous results derive (15.5)

1°) Multiply (15.1) by x and integrate:

$$x.(15.1) \Rightarrow x.\frac{\partial}{\partial t}\nabla^2\psi + x.\frac{\partial}{\partial t}\left(\frac{\psi}{Ld^2}\right) + x.\,J(\psi,\nabla^2\psi) + \beta.x.\frac{\partial\psi}{\partial x} = 0$$

2°)

$$* \int_{-\infty}^{+\infty} x\frac{\partial}{\partial t}\nabla^2\psi\, dS = \int_{-\infty}^{+\infty}\left[\nabla.\left(x.\frac{\partial}{\partial t}\nabla\psi\right) - \nabla x.\frac{\partial}{\partial t}\nabla\psi\right]dS$$

$$= \underbrace{\int_{-\infty}^{+\infty}\nabla.\left(x\frac{\partial}{\partial t}\nabla\psi\right)dS}_{=0 \text{ because Gauss Theorem}} - \int_{-\infty}^{+\infty}\nabla x.\frac{\partial}{\partial t}\nabla\psi\, dS$$

$$= -\int_{-\infty}^{+\infty}\nabla x.\nabla\left(\frac{\partial}{\partial t}\psi\right)dS$$

$$= -\int_{-\infty}^{+\infty}\begin{vmatrix}\frac{\partial x}{\partial x}\\ 0\\ 0\end{vmatrix}\cdot\begin{vmatrix}\frac{\partial^2\psi}{\partial x\partial t}\\ \cdot\\ \cdot\end{vmatrix}dS$$

$$= -\int_{-\infty}^{+\infty}\frac{\partial^2\psi}{\partial x\partial t}\,dx\,dy$$

$$= -\int_{-\infty}^{+\infty} d\left(\frac{\partial\psi}{\partial t}\right)dy$$

$= 0$ because integration of y between $-\infty$ and $+\infty$

$$* \quad \frac{1}{Ld^2}\int_{-\infty}^{+\infty} x\frac{\partial \psi}{\partial t}\, dS = \frac{1}{Ld^2}\frac{d}{dt}\int_{-\infty}^{+\infty} x\,\psi\, dS \quad \text{because} \quad ?$$

$$* \quad \int_{-\infty}^{+\infty} \beta x \frac{\partial \psi}{\partial x}\, dS = \beta\int_{-\infty}^{+\infty}\left[\frac{\partial}{\partial x}(x\psi) - \psi\right] dS$$

$$= \beta\int_{-\infty}^{+\infty}\frac{\partial}{\partial x}(x\psi)\, dx\,dy - \beta\int \psi\, dS$$

$$= \underbrace{\cancel{\beta\int_{-\infty}^{+\infty} d(x\psi)\, dy}}_{=0 \text{ because } y \text{ integration}} - \beta\int_{-\infty}^{+\infty}\psi\, dS$$

$$* \quad \int_{-\infty}^{+\infty} x\, J(\psi, \nabla^2\psi)\, dS \underset{J \text{ property}}{=} \int_{-\infty}^{+\infty} \nabla^2\psi\, J(x,\psi)\, dS$$

$$= \int_{-\infty}^{+\infty} \nabla^2\psi \frac{\partial \psi}{\partial y}\, dS$$

$$= \underbrace{\cancel{\int_{-\infty}^{+\infty} \nabla\cdot\left(\nabla\psi \frac{\partial\psi}{\partial y}\right)}}_{=0} - \nabla\psi\cdot\nabla\left(\frac{\partial\psi}{\partial y}\right) dS$$

because of Gauss Theorem

$$= -\int_{-\infty}^{+\infty} \nabla\psi\cdot\nabla\left(\frac{\partial \psi}{\partial y}\right) dS$$

$$= -\int_{-\infty}^{+\infty} \nabla\psi\cdot\frac{\partial}{\partial y}(\nabla\psi)\, dS$$

$$= -\int_{-\infty}^{+\infty} \frac{\partial}{\partial y}\left(\frac{|\nabla\psi|^2}{2}\right) dS$$

$$= -\int_{-\infty}^{+\infty} d\left(\frac{|\nabla\psi|^2}{2}\right) dx$$

$= 0$ because of x integration

3°) Collect each term:

$$\Rightarrow \quad \frac{d}{dt}\int_{-\infty}^{+\infty} \frac{x\psi}{Ld^2}\, dS = -\int_{-\infty}^{+\infty} \beta\psi\, dS$$

$$\Leftrightarrow \quad \boxed{\frac{d}{dt}\int_{-\infty}^{+\infty} x\psi\, dS = -\beta Ld^2 \int_{-\infty}^{+\infty} \psi\, dS} \qquad (15.6)$$

$$\text{So} \quad \frac{d}{dt}\left(\frac{\int_{-\infty}^{+\infty} x\psi\, dS}{\int_{-\infty}^{+\infty} \psi\, dS}\right) = \frac{\left(\frac{d}{dt}\int_{-\infty}^{+\infty} x\psi\, dS\right)\int_{-\infty}^{+\infty}\psi\, dS - 0}{\left(\int \psi\, dS\right)^2}$$

because $\frac{d}{dt}\int_{-\infty}^{+\infty} \psi\, dS = 0$ ⊛

⊛ this is in fact equivalent to the integration of (15.1) (15.1: $\frac{D}{Dt}(\psi) = 0$)

$$= \frac{\left(\frac{d}{dt}\int_{-\infty}^{+\infty} x\psi\, dS\right)}{\left(\int_{-\infty}^{+\infty} \psi\, dS\right)^2}$$

$$= \frac{-\beta Ld^2 \cancel{\int_{-\infty}^{+\infty} \psi\, dS}}{\cancel{\int_{-\infty}^{+\infty} \psi\, dS}} \quad \text{with (15.6)}$$

$$= -\beta Ld^2$$

4°) Multiply $\frac{d}{dt}\int_{-\infty}^{+\infty} \psi\, dS = 0$ by y, we get

$$\frac{d}{dt}\int_{-\infty}^{+\infty} y\psi\, dS = 0 \qquad (15.7)$$

$$\text{And} \quad \frac{d\bar{y}}{dt} = \frac{\left(\frac{d}{dt}\int_{-\infty}^{+\infty} y\psi\, dS\right)\int_{-\infty}^{+\infty}\psi\, dS - \cancel{\int_{-\infty}^{+\infty} y\psi\, dS \overbrace{\left(\frac{d}{dt}\int \psi\, dS\right)}^{0}}}{\left(\int_{-\infty}^{+\infty} \psi\, dS\right)^2}$$

$$= \frac{\frac{d}{dt}\int_{-\infty}^{+\infty} y\psi\, dS}{\int_{-\infty}^{+\infty} \psi\, dS}$$

$$= 0 \quad \text{because of (15.7)}$$