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Variational problem for vortices attached to seamounts

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Abstract

The existence of an energy maximizer relative to a class of rearrangements of a given function is proved. The maximizers are stationary and stable solutions of the two-dimensional barotropic vorticity equation, governing the evolution of geophysical flow over a surface of variable height. The theorem proved implies the existence of a family of stable vortices with anticyclonic potential vorticity over a seamount, and a similar family of vortices with cyclonic potential vorticity over a localized depression.

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1. Introduction

In this paper, we will prove the existence of maximizers for a variational problem associated with geophysical flows over a surface of variable height (e.g. a seamount in the ocean or a mountain in the atmosphere). The basic equation governing such flows is the two-dimensional barotropic vorticity equation

$$\frac{\partial \zeta}{\partial t} + J(\psi, \zeta) = 0. \tag{1}$$

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Here ψ is the stream function, J denotes the Jacobian, and ζ is given by

$$-\zeta = \Delta\psi + h,$$

where h is the height of the bottom surface relative to some reference value. Usually $-\zeta$ is called the *potential vorticity* (PV), with positive ζ corresponding to anticyclonic PV.

According to Eq. (1), ζ is advected by an incompressible velocity field. Thus, $\zeta(x,t)$ remains in the set of rearrangements of the initial condition $\zeta(x,0)$. The dynamics defined by Eq. (1) also conserves the energy (to be defined below). A field ζ that maximizes the energy in a set of rearrangements therefore corresponds to a stationary and stable flow, and by proving the existence of a maximizer one proves the existence of such a flow.

This variational principle was used by Benjamin in a theory of three-dimensional vortex rings [1]. It has also been used to prove rigorously the existence of families of three-dimensional vortex rings [2], two-dimensional vortex couples [3], and stationary and stable vortices in two-dimensional shear flow [9] and three-dimensional quasi-geostrophic shear flow [5]. Here, it will be used to prove the existence of a family of stationary and stable vortices with anticyclonic PV attached to a localized seamount or mountain. By symmetry, the corresponding result holds for vortices with cyclonic PV attached to a localized depression. Note that the shape of the topographic anomaly h is arbitrary, as long as it has compact support and the same sign everywhere.

Heuristic arguments for the theorem to be proved here, as well as numerical simulations illustrating it, were recently presented by Nycander and LaCasce [11]. They also gave heuristic arguments for the existence of a distinct class of minimum energy flows; these will not be considered here. Heuristic arguments for the extension of the present proof to three-dimensional quasigeostrophic flow have also recently been presented [10].

2. Notation and statement of the main result

Henceforth we assume $p \in (2,\infty)$. The ball centered at $x \in \mathbb{R}^2$ with radius R is denoted $B_R(x)$; in particular when the center is the origin we write B_R. For a measurable set $E \subset \mathbb{R}^2$, $|E|$ denotes the two-dimensional Lebesgue measure of E. If E is measurable, then $x \in E$ is called a density point of E whenever

$$|B_\varepsilon(x) \cap E| > 0$$

for all positive ε. The set of all density points of E is denoted $\mathrm{den}(E)$.

For a measurable function $\zeta : \mathbb{R}^2 \to \mathbb{R}$, the *strong support* or simply the *support* of ζ denoted $\mathrm{supp}(\zeta)$ is defined

$$\mathrm{supp}(\zeta) = \{x : \zeta(x) > 0\}.$$

As usual $\|\zeta\|_{r,D}$ denotes the L^r-norm over the set D. If f and g are non-negative measurable functions that vanish outside sets of finite measure in $\mathbb{R}^2$, we say f is a *rearrangement of* g whenever

$$|\{x \in \mathbb{R}^2 : f(x) \geqslant \alpha\}| = |\{x \in \mathbb{R}^2 : g(x) \geqslant \alpha\}|$$

for every positive α. Let us fix $\zeta_0 \in L^p(\mathbb{R}^2)$ to be a non-negative function vanishing outside a set of measure πa^2, for some positive a. For simplicity we assume that $\|\zeta_0\|_1 = 1$. The set of all rearrangements of ζ_0 on $\mathbb{R}^2$ which vanish outside bounded sets is denoted $\mathscr{F}$. The subset of $\mathscr{F}$ comprising functions vanishing outside the ball B_R is denoted $\mathscr{F}(R)$; henceforth we assume $R > a$ in order to ensure $\mathscr{F}(R)$ is non-empty.

For a non-negative $\zeta \in L^p(\mathbb{R}^2)$ having bounded support, we define the *energy functional*

$$\Psi(\zeta) = \frac{1}{2}\int_{\mathbb{R}^2} \zeta K\zeta + \int_{\mathbb{R}^2} \eta\zeta,$$

where

$$K\zeta(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \log\frac{1}{|x-y|}\,\zeta(y)\,\mathrm{d}y$$

and $\eta = Kh$; here $h \in L^p(\mathbb{R}^2)$ is a non-negative and non-trivial function with compact support. Let B_{r_h} be the smallest ball containing supp(h); we assume that $r_h > a$. Let us consider the following variational problem

$$(P): \sup_{\zeta\in\mathscr{F}} \Psi(\zeta).$$

The solution set for (P) is denoted Σ. We are now ready to state the main result of this paper.

Theorem 1. *For the variational problem (P), Σ is not empty; moreover, if $\hat{\zeta} \in \Sigma$ and we set $\psi = K\hat{\zeta} + \eta$, then ψ satisfies the following non-autonomous semilinear elliptic partial differential equation*:

$$-\Delta\psi = \phi\circ\psi + h, \tag{2}$$

a.e. in $\mathbb{R}^2$, for some increasing function ϕ unknown a priori.

Note that from Eq. (2) it follows that $\hat{\zeta}$ and ψ are functionally dependent, hence their Jacobian vanishes, formally speaking. Therefore Eq. (2) is the equation for stationary solutions of Eq. (1).

Also note that when h and ζ_0 have opposite signs the supremum in the variational problem (P) is infinite. Indeed when h is a negative function, η behaves like $\log|x|$ for large $|x|$. Therefore, letting $\mathscr{L}$ denote the linear part of Ψ, and letting $t_n \in \mathbb{R}^2$ be a sequence like $(n, 0)$, $\mathscr{L}(\zeta_0(\cdot - t_n))$ tends to infinity as $n \to \infty$.

3. Preliminaries

We begin with the following useful estimate.

Lemma 1. *Suppose $f \in L^p(\mathbb{R}^2)$ is a non-negative function with compact support. Then*

$$Kf(x) \leqslant C\|f\|_p. \tag{3}$$

Proof. We begin with the simple observation that

$$Kf(x) \leqslant \frac{1}{2\pi}\int_{B_1(x)} \log\frac{1}{|x-y|}\, f(y)\,\mathrm{d}y \leqslant \frac{1}{2\pi}\int_{B_1(x)} \log\frac{1}{|x-y|}\, \tilde{f}(y)\,\mathrm{d}y,$$

where $\tilde{f}$ is the Schwarz decreasing rearrangement of f with respect to x; note that the second inequality is a consequence of a standard rearrangement inequality. Now by an application of Hölder's inequality we derive (3), where

$$C = \left(\frac{1}{2\pi}\right)^{1/p}\left(\int_0^1 r\log\frac{1}{r}\,\mathrm{d}r\right)^{1/q}.$$

Here q is the conjugate exponent of p; that is, $1/p+1/q=1$. □

The following has been proved in [6].

Lemma 2. *Let $q \geqslant 1$ and let U be a bounded open subset of $\mathbb{R}^2$. Then*

$$K: L^p(U) \to L^q(U)$$

is a compact linear operator; the compactness is in the sense that if $\{\zeta_n\}$ is a sequence of functions, bounded in $L^p(\mathbb{R}^2)$ and vanishing outside U, then the restrictions to U of the $K\zeta_n$ have a subsequence converging in the L^q-norm. Moreover: if $\zeta \in L^p(\mathbb{R}^2)$ vanishes outside U, then $K\zeta \in W^{2,p}_{\mathrm{loc}}(\mathbb{R}^2)$ and verifies the following Poisson equation:

$$-\Delta u = \zeta,$$

almost everywhere in $\mathbb{R}^2$.

The next result has been proved in [2].

Lemma 3. *If $\overline{\mathscr{F}(R)}^{\mathrm{w}}$ denotes the weak closure of $\mathscr{F}(R)$* in *$L^p(B_R)$, then $\overline{\mathscr{F}(R)}^{\mathrm{w}}$ is convex and weakly sequentially compact.*

In order to prove the existence part of Theorem 1 we first consider the following family of variational problems:

$$(P_R):\ \sup_{\zeta\in\mathscr{F}(R)} \Psi(\zeta).$$

The solution set of (P_R) is denoted Σ_R. We show that (P_R) are solvable. To do this we need the following result, which is a simple variation of Lemma 2.15 in [4].

Lemma 4. *Let q be the conjugate exponent of p. Let $g \in L^q(B_R)$ and denote by $L_\alpha(g)$ the level set of g at height α; that is,*

$$L_\alpha(g) = \{x \in B_R : g(x) = \alpha\}.$$

Let $\mathscr{I}: L^p(B_R) \to \mathbb{R}$ be the linear functional defined by

$$\mathscr{I}(\zeta) = \int_{B_R} \zeta g.$$

If $\hat{\zeta}$ is a maximizer of $\mathscr{I}$ relative to $\overline{\mathscr{F}(R)}^{\rm w}$ and if

$$|L_\alpha(g) \cap \operatorname{supp}(\hat{\zeta})| = 0$$

for every $\alpha \in \mathbb{R}$, then $\hat{\zeta} \in \mathscr{F}(R)$ and

$$\hat{\zeta} = \phi_R \circ g,$$

almost everywhere in B_R, for some increasing function ϕ_R.

Remark. In Lemma 4, by redefining $\hat{\zeta}$ on a set of zero measure in B_R if necessary, we can make the conclusion of the lemma to hold everywhere in B_R.

Lemma 5. *The variational problems (P_R) are solvable and if $\hat{\zeta}_R \in \Sigma_R$, then*

$$\hat{\zeta}_R = \phi_R \circ (K\hat{\zeta}_R + \eta),$$

almost everywhere in B_R, for some increasing function ϕ_R.

Proof. By Lemma 2 we have

$$-\Delta\eta = h.$$

Hence, using elliptic regularity theory it follows that $\eta \in W^{2,p}_{\rm loc}(\mathbb{R}^2)$. Therefore $\eta \in C(\mathbb{R}^2)$, by the Sobolev embedding theorem. Let us note in passing that η is negative at long range points; this follows from

$$\eta(x) \leqslant \frac{1}{2\pi}\,\|h\|_1 \log\frac{2}{|x|} \qquad (4)$$

provided $\operatorname{dist}(x, \operatorname{supp}(h)) > |x|/2$. Thus η is bounded from above, so we define

$$\eta_\infty = \sup_{\mathbb{R}^2} \eta.$$

Note that Ψ is the summation of a quadratic and a linear functional; that is,

$$\Psi = \mathscr{Q} + \mathscr{L}.$$

By Lemma 2, K is compact. Thus $\mathscr{Q}$ is weakly sequentially continuous. Also since η is continuous it follows that $\mathscr{L}$ is also weakly sequentially continuous; this proves that Ψ is weakly sequentially continuous on $L^p(B_R)$. Since $\overline{\mathscr{F}(R)}^{\rm w}$ is weakly sequentially compact, by Lemma 3, it follows that Ψ has a maximum relative to $\overline{\mathscr{F}(R)}^{\rm w}$, say $\bar{\zeta}$. For any $\zeta \in \overline{\mathscr{F}(R)}^{\rm w}$ and any $t \in [0,1]$, $\bar{\zeta} + t(\zeta - \bar{\zeta}) \in \overline{\mathscr{F}(R)}^{\rm w}$, by Lemma 3. Using the first variation of Ψ at $\bar{\zeta}$ we obtain

$$\Psi(\bar{\zeta} + t(\zeta - \bar{\zeta})) - \Psi(\bar{\zeta}) = t\langle \Psi'(\bar{\zeta}), \zeta - \bar{\zeta}\rangle + o(t),$$

as $t \to 0^+$; here $\langle\cdot,\cdot\rangle$ stands for the pairing between $L^p(B_R)$ and its dual, and $\Psi'(\cdot)$ stands for the derivative. Since $\bar{\zeta}$ is a maximizer it follows that

$$\langle \Psi'(\bar{\zeta}), \zeta - \bar{\zeta}\rangle \leqslant 0.$$

Therefore, $\bar{\zeta}$ is a maximizer for the linear functional

$$\langle \Psi'(\bar{\zeta}), \cdot\rangle,$$

relative to $\overline{\mathscr{F}(R)^{\mathrm{w}}}$. Since $\Psi'(\bar{\zeta})$ can be identified with $K\bar{\zeta}+\eta\in L^q(B_R)$, where q denotes the conjugate of p, it follows that $\bar{\zeta}$ is a maximizer of

$$\int_{B_R}\zeta\ \left(K\bar{\zeta}+\eta\right)$$

relative to $\zeta\in\overline{\mathscr{F}(R)^{\mathrm{w}}}$. From Lemma 2 we obtain

$$-\Delta\left(K\bar{\zeta}+\eta\right)=\bar{\zeta}+h.$$

Therefore, the level sets of $K\bar{\zeta}+\eta$ over $\mathrm{supp}(\bar{\zeta})$ are negligible, by for example Lemma 7.7 of [7] . Thus, we can apply Lemma 4 to deduce that $\bar{\zeta}\in\mathscr{F}(R)$ and

$$\bar{\zeta}=\phi\circ\left(K\bar{\zeta}+\eta\right),$$

almost everywhere in B_R, for some increasing function ϕ. Therefore $\bar{\zeta}\in\Sigma_R$.

Now consider $\hat{\zeta}\in\Sigma_R$. Since Ψ is weakly sequentially continuous it follows that $\hat{\zeta}$ maximizes Ψ relative to $\overline{\mathscr{F}(R)^{\mathrm{w}}}$. Now by applying the first variation argument above we can similarly prove existence of an increasing function ϕ_R such that

$$\hat{\zeta}=\phi_R\circ\left(K\hat{\zeta}+\eta\right),$$

almost everywhere in B_R. □

4. Some more lemmas

The following result is crucial.

Lemma 6. *Suppose that $\beta_0\in(0,1)$. Then there exists $r(\beta_0)>r_h$ such that if $R>r_h$ and $\hat{\zeta}_R\in\Sigma_R$, then*

$$\|\hat{\zeta}_R\|_{1,B_{r(\beta_0)}}\geqslant\beta_0. \tag{5}$$

Proof. Let us recall that $\|\zeta_0\|_1=1$. Fix $r>r_h$. Assume that $R>r$ and $\hat{\zeta}_R\in\Sigma_R$. Thus,

$$\int_{\mathbb{R}^2}\hat{\zeta}_R\eta=\int_{B_r}\hat{\zeta}_R\eta+\int_{B_R\setminus B_r}\hat{\zeta}_R\eta. \tag{6}$$

We now estimate the two integrals on the right-hand side of (6). Since $\eta\leqslant\eta_\infty$ we have

$$\int_{B_r}\hat{\zeta}_R\eta\leqslant\eta_\infty\|\hat{\zeta}_R\|_{1,B_r}. \tag{7}$$

Also, for $x\in B_R\setminus B_r$ we have

$$\eta(x)\leqslant\frac{1}{2\pi}\|h\|_1\log\frac{1}{r-r_h}.$$

Therefore,

$$\int_{B_R\setminus B_r} \hat{\zeta}_R \eta \leqslant \frac{1}{2\pi}\|h\|_1\left(\log\frac{1}{r-r_h}\right)\|\hat{\zeta}_R\|_{1,B_R\setminus B_r}$$

$$= \frac{1}{2\pi}\|h\|_1\left(\log\frac{1}{r-r_h}\right)\left(1-\|\hat{\zeta}_R\|_{1,B_r}\right), \tag{8}$$

where we have used $\|\zeta_0\|_1 = 1$. Using (7) and (8) we derive

$$\mathscr{L}\left(\hat{\zeta}_R\right) \leqslant \eta_\infty\|\hat{\zeta}_R\|_{1,B_r} + \frac{1}{2\pi}\|h\|_1\left(1-\|\hat{\zeta}_R\|_{1,B_r}\right)\log\frac{1}{r-r_h}. \tag{9}$$

It follows from Riesz's inequality, see, for example, [8], that

$$\mathscr{Q}\left(\hat{\zeta}_R\right) \leqslant \mathscr{Q}(\zeta_0^*), \tag{10}$$

where ζ_0^* denotes the Schwarz rearrangement of ζ_0 about the origin. Since $\hat{\zeta}_R$ is a maximizer we have

$$\mathscr{Q}(\zeta_0^*) + \mathscr{L}(\zeta_0^*) = \Psi(\zeta_0^*) \leqslant \mathscr{Q}\left(\hat{\zeta}_R\right) + \mathscr{L}\left(\hat{\zeta}_R\right).$$

We can now use (9) and (10) to derive

$$\mathscr{L}(\zeta_0^*) \leqslant \eta_\infty\|\hat{\zeta}_R\|_{1,B_r} + \frac{1}{2\pi}\|h\|_1\left(1-\|\hat{\zeta}_R\|_{1,B_r}\right)\log\frac{1}{r-r_h}. \tag{11}$$

Now let $r(\beta_0)$ be the solution of the following equation:

$$\frac{1}{2\pi}\|h\|_1\log\frac{1}{r-r_h} = \frac{\mathscr{L}(\zeta_0^*) - \beta_0\eta_\infty}{1-\beta_0}.$$

Then from (11), replacing r by $r(\beta_0)$, we derive

$$(\|\hat{\zeta}_R\|_{1,B_{r(\beta_0)}} - \beta_0)(\eta_\infty - \mathscr{L}(\zeta_0^*)) \geqslant 0. \tag{12}$$

But $\mathscr{L}(\zeta_0^*) = \int \zeta_0^*\eta < \eta_\infty$, since $\|\zeta_0\|_1 = 1$ and h is non-trivial. Thus, the second factor on the left-hand side of (12) is non-negative, hence we derive (5). □

Lemma 7. *There exists R^* such that if $R > R^*$ and $\hat{\zeta}_R \in \Sigma_R$, then*

$$\operatorname{supp}\left(\hat{\zeta}_R\right) \subset B_{R^*}, \tag{13}$$

modulo a set of zero measure.

Proof. Suppose the assertion is false. Then there exist sequences $(R_n), (x_n)$ and $\left(\hat{\zeta}_{R_n}\right) \equiv \left(\hat{\zeta}_n\right)$ such that:

(1) $R_n \to \infty$.
(2) $\hat{\zeta}_n \in \Sigma_{R_n}$.
(3) $x_n \in \operatorname{den}\left(\operatorname{supp}\left(\hat{\zeta}_n\right)\right)$ and $\|x_n\|_{\mathbb{R}^2} \to \infty$, where $\|\cdot\|_{\mathbb{R}^2}$ denotes the usual Euclidean norm in $\mathbb{R}^2$.

Without loss of generality we may assume that $\|x_n\|_{\mathbb{R}^2} = R_n$ and (R_n) is increasing; moreover we may assume that $R_n \geqslant r^*$, where $r^* \equiv r\left(\frac{3}{4}\right)$, see Lemma 6. Let $\psi_n = K\hat{\zeta}_n + \eta$. We now proceed to bound $\psi_n(x_n)$ from above.

$$K\hat{\zeta}_n(x_n) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \log\frac{1}{|x_n - y|}\,\hat{\zeta}_n(y)\,\mathrm{d}y$$
$$= \frac{1}{2\pi}\int_{B_{r^*}} \log\frac{1}{|x_n - y|}\,\hat{\zeta}_n(y)\,\mathrm{d}y + \frac{1}{2\pi}\int_{\mathbb{R}^2\setminus B_{r^*}} \log\frac{1}{|x_n - y|}\,\hat{\zeta}_n(y)\,\mathrm{d}y$$
$$\leqslant \frac{1}{2\pi}\|\hat{\zeta}_n\|_{1,B_{r^*}} \log\frac{1}{R_n - r^*} + C, \tag{14}$$

where in the last inequality we have used Lemma 1. Also

$$\eta(x_n) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \log\frac{1}{|x_n - y|}\,h(y)\,\mathrm{d}y \leqslant \frac{1}{2\pi}\|h\|_1 \log\frac{1}{R_n - r^*}. \tag{15}$$

From (14) and (15) we obtain

$$\psi_n(x_n) \leqslant C + \frac{1}{2\pi}\|\hat{\zeta}_n\|_{1,B_{r^*}} \log\frac{1}{R_n - r^*} + \frac{1}{2\pi}\|h\|_1 \log\frac{1}{R_n - r^*}. \tag{16}$$

To derive a contradiction we first note that there exists a sequence (y_n) in B_{r^*} such that $y_n \notin \operatorname{supp}(\hat{\zeta}_n)$. Now, we estimate $\psi_n(y_n)$ from below.

$$K\hat{\zeta}_n(y_n) = \frac{1}{2\pi}\int_{B_{r^*}} \log\frac{1}{|y_n - y|}\,\hat{\zeta}_n(y)\,\mathrm{d}y + \frac{1}{2\pi}\int_{\mathbb{R}^2\setminus B_{r^*}} \log\frac{1}{|y_n - y|}\,\hat{\zeta}_n(y)\,\mathrm{d}y$$
$$\geqslant \frac{1}{2\pi}\|\hat{\zeta}_n\|_{1,B_{r^*}} \log\frac{1}{2r^*} + \frac{1}{2\pi}\left(1 - \|\hat{\zeta}_n\|_{1,B_{r^*}}\right) \log\frac{1}{R_n + r^*}. \tag{17}$$

Also

$$\eta(y_n) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \log\frac{1}{y_n - y}\,h(y)\,\mathrm{d}y \geqslant \frac{1}{2\pi}\|h\|_1 \log\frac{1}{2r^*}. \tag{18}$$

Therefore, from (17) and (18) we find

$$\psi_n(y_n) \geqslant \frac{1}{2\pi}\|\hat{\zeta}_n\|_{1,B_{r^*}} \log\frac{1}{2r^*} + \frac{1}{2\pi}\left(1 - \|\hat{\zeta}_n\|_{1,B_{r^*}}\right) \log\frac{1}{R_n + r^*}$$
$$+ \frac{1}{2\pi}\|h\|_1 \log\frac{1}{2r^*}. \tag{19}$$

Therefore, from (16) and (19) we derive

$$\psi_n(x_n) - \psi_n(y_n) \leqslant C + \frac{1}{2\pi}\|\hat{\zeta}_n\|_{1,B_{r^*}} \log\frac{1}{R_n - r^*} + \frac{1}{2\pi}\|h\|_1 \log\frac{1}{R_n - r^*}$$
$$- \frac{1}{2\pi}\|\hat{\zeta}_n\|_{1,B_{r^*}} \log\frac{1}{2r^*} - \frac{1}{2\pi}\left(1 - \|\hat{\zeta}_n\|_{1,B_{r^*}}\right) \log\frac{1}{R_n + r^*}$$
$$- \frac{1}{2\pi}\|h\|_1 \log\frac{1}{2r^*}. \tag{20}$$

From Lemma 6 we have

$$\|\hat{\zeta}_n\|_{1,B_{r^*}} \geqslant \tfrac{3}{4}. \tag{21}$$

Note that when n is sufficiently large, we obtain from (20) and (21)

$$\psi_n(x_n) - \psi_n(y_n) \leqslant C + \frac{1}{2\pi}\,\|\hat{\zeta}_n\|_{1,B_{r^*}} \log\frac{1}{R_n - r^*} - \frac{1}{2\pi}\left(1 - \|\hat{\zeta}_n\|_{1,B_{r^*}}\right)\log\frac{1}{R_n + r^*}. \tag{22}$$

We can now infer from (21) and (22) the existence of $n^* \in \mathbb{N}$ such that

$$\psi_{n^*}(x_{n^*}) - \psi_{n^*}(y_{n^*}) < 0. \tag{23}$$

However from Lemma 4, and the Remark after it, there exists ϕ_{n^*}, an increasing function, such that

$$\hat{\zeta}_{n^*} = \phi_{n^*} \circ \psi_{n^*},$$

everywhere in $B_{R_{n^*}}$. Therefore, ψ_{n^*} attains its largest values over $\mathrm{den}\left(\mathrm{supp}\left(\hat{\zeta}_{n^*}\right)\right)$, so (23) is false. Hence we are done. □

5. Proof of Theorem 1

Let R^* be as in Lemma 7. Then Σ_{R^*} is not empty and obviously

$$\Sigma_{R^*} \subset \Sigma,$$

which proves the existence part of Theorem 1. To derive (2), consider $\hat{\zeta} \in \Sigma$. Since $\hat{\zeta}$ has compact support, there would exist $\bar{R}$ such that

$$\mathrm{supp}\,\hat{\zeta} \subset B_{\bar{R}},$$

modulo a set of zero measure. Therefore, by Lemma 5 there exists an increasing function $\bar{\phi}$ such that

$$\hat{\zeta} = \bar{\phi} \circ \psi, \tag{24}$$

almost everywhere in $B_{\bar{R}}$, where $\psi = K\hat{\zeta} + \eta$. We now proceed to modify (24), in order to find a similar equation which holds throughout $\mathbb{R}^2$. From (24) we infer the existence of a constant γ for which

$$\mathrm{supp}\left(\hat{\zeta}\right) = \{x \in B_{\bar{R}} \,|\, \psi \geqslant \gamma\}, \tag{25}$$

modulo a set of zero measure. From (4) we obtain the following bound:

$$\psi(x) \leqslant \frac{1}{2\pi}\,\|\hat{\zeta} + h\|_1 \log\frac{2}{|x|}$$

provided $\mathrm{dist}\left(x, \mathrm{supp}\left(\hat{\zeta} + h\right)\right) \geqslant |x|/2$, hence there exists $R' > \bar{R}$ such that

$$\psi(x) < \gamma - 1 \tag{26}$$

provided $x \in \mathbb{R}^2 \setminus B_{R'}$. Also, since $\hat{\zeta} \in \Sigma_{R'}$ we can apply Lemma 5 once again to deduce the existence of another increasing function, say ϕ', such that

$$\hat{\zeta} = \phi' \circ \psi, \tag{27}$$

almost everywhere in $B_{R'}$. We now define

$$\phi(t) = \begin{cases} \phi'(t), & t \geqslant \gamma, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore by applying (25)–(27) we obtain

$$\hat{\zeta} = \phi \circ \psi, \tag{28}$$

almost everywhere in $\mathbb{R}^2$. Now using Lemma 2 and the fact that

$$-\Delta\eta = h,$$

almost everywhere in $\mathbb{R}^2$, we derive (2), so we are done. □

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