Wind-driven and Topographically Steered Ocean Circulation in an Equatorial Basin

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May 28, 2013
Abstract

In the Nordic Seas and Arctic Ocean the large-scale ocean circulation traces out the underlying bathymetry. The theory behind the circulation in these high latitude basins is one of circulation along closed contours of $\frac{fH}{\Pi}$, where $f$ is the Coriolis parameter and $H$ is the depth. If the equator were to cross a basin with continental slopes, the contours of $\frac{fH}{\Pi}$ would converge and meet at the intersection of the western boundary and the equator, where both the Coriolis parameter and the depth go to zero. The steady circulation induced by a wind north of the Equator under these circumstances is investigated using both analytical and numerical analysis, with emphasis on a possible southern hemisphere response and frictional boundary layer.

A nonlinear shallow water model is used to simulate the circulation in the equatorial basin. No steady circulation in the south hemisphere is found to be induced by the wind north of the equator. A frictional boundary layer forms in the vicinity of the intersection of the western boundary and the equator, where the friction enables cross contour flow, resulting in a closed circulation. This return flow is along contours of $\frac{f}{\Pi}$ with lower values of the Coriolis parameter, but are still north of the equator.

A complete analytical solution for the circulation is not within the scope of this study, but under certain assumptions a solution for the boundary layer is presented. This solution suggests boundary layer length scales that differ on the parametrization of the friction. The solutions for two different linear parametrizations, where one takes the varying depth of the basin into account and one does not, are exponentially decreasing with $r^2$ and $r$, respectively, where $r$ is the radial distance from the point where the contours of $\frac{f}{\Pi}$ converge. Similar exponential behaviour is found in the numerically simulated solutions, indicating reliability in the presented analytical solutions.
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1 Introduction

In the middle of the 20th century, Sverdrup (1947) presented a theory for the depth-integrated large-scale wind-driven ocean circulation. It assumes a flat bottom, so that the vertical velocity at the bottom is zero, and frictional dissipation is neglected. What is now referred to as the Sverdrup relation, relates the transport to the curl of the wind stress according to

\[ V = \frac{1}{\beta} \nabla \times \tau^w, \quad (1) \]

where \( V \) is the vertically integrated meridional transport, \( \beta \) is the meridional gradient of the Coriolis parameter and \( \tau^w \) is the wind stress. It states that the depth-integrated meridional transport is proportional to the curl of the wind stress. If the sign of the curl in the Sverdrup relation is the same in the whole domain, conservation of mass requires a return flow. The models of Stommel (1948) and Munk (1950) give a frictional boundary layer which results in a western intensification, or western boundary current, that closes the circulation; see figure 1. This is a cornerstone in dynamical oceanography, which describes the interior circulation and western boundary currents in the major ocean basins.

Figure 1: Example of the Stommel model, with a western intensification of the large-scale circulation. (Vallis, 2006)
In the Nordic Seas and Arctic Ocean, however, it has long been recognized that the bottom topography plays an important role (e.g. Helland-Hansen and Nansen, 1909). These high-latitude basins have weak stratification and both surface and bottom currents have been found to trace out the bathymetry (e.g. Woodgate et al., 2001; Poulain et al., 1996). A leading order balance for the circulation in these northern basins, which will be more thoroughly described in section 2, is that of a flow following the contours of \( \frac{f}{H} \), where \( f = f(y) \) is the Coriolis parameter and \( H = H(x, y) \) is the depth.

The Sverdrup circulation, and the western boundary currents of Munk and Stommel, are based on the assumption of constant depth. Disturbances in the form of long Rossby waves then travel westward along latitudes, or contours of \( f \), which intersect the coasts. These are called blocked contours.

Figure 2: The near-surface circulation of the Norwegian Sea, according to the observations by Helland-Hansen and Nansen (1909).
2 Theory

In the Nordic Seas and Arctic Ocean the topography dominates the contours of \( \frac{f}{H} \) which results in contours that can close on themselves; closed contours.

In a basin with predominant closed contours of \( \frac{f}{H} \), a localized wind stress can remotely control the circulation through the whole basin. However, if this basin were to lie on the Equator, the lines of \( \frac{f}{H} \) no longer close on themselves, despite that the depth contours do; \( f \) must go to zero at the Equator and the depth must go to zero at the coasts. The dynamics of the wind-driven flow under these circumstances will be the essence of this study, in which the following questions are aimed to be answered:

- What are the characteristics of the induced large-scale steady circulation?
- Will there be frictional boundary layers and intense western boundary currents near the equator?
- Can a localized wind north of the Equator induce a steady circulation in the southern hemisphere?

The answers are sought under simplified conditions; a depth-integrated shallow water with a linear bottom friction and linear continental slopes. The analysis has both a linear analytical approach and a numerical approach where a nonlinear model is used.

The rest of the report will be organized as follows: Section 2 on the theory behind flow along contours of \( \frac{f}{H} \), section 3 on the theoretical analysis of the equatorial basin, section 4 on the numerical analysis and section 5 on the main conclusions.

2.1 Closed contours

To present the theory of a flow that to the leading order is along closed contours of \( \frac{f}{H} \), we will compare the scales of the velocity components parallel and normal to the contours. We start with a barotropic, linear, steady state
version of the shallow water equations,

\[ f k \times v = -g \nabla \eta + \frac{\partial \tau}{\partial z}, \]  \hspace{1cm} (2)

where \( f = f(y) \) is the Coriolis parameter, \( v \) is the horizontal velocity vector, \( g \) is the constant gravitational acceleration, \( \eta \) is the free surface and \( \tau = (\tau_x, \tau_y) \) is a horizontal stress. \( k \) is the vertical unit vector and \( \nabla = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right) \) is the horizontal gradient operator. With a bottom topography \( H = H(x,y) \) and a \( \beta \)-plane where \( f = f_0 + \beta y \), the depth-integrated equation (2) is

\[ f k \times V = -gH \nabla \eta + \tau_w - \tau_b \]  \hspace{1cm} (3)

where \( V = vH \) is the depth-integrated horizontal velocity and the forcing is written as the components at the bottom and top of the basin; a bottom friction \( \tau_b \) and the wind stress \( \tau_w \). Together with the continuity equation,

\[ \nabla \cdot (HV) = \nabla \cdot V = 0, \]  \hspace{1cm} (4)

equation (3) forms a depth-integrated shallow water system.

Equation (3) integrated along a closed curve \( C \), on which \( f_H \) is constant and to which the line segment \( ds \) is parallel,

\[ \oint_C k \times V \cdot ds = -H \oint_C g \nabla \eta \cdot ds + \oint_C \frac{1}{f} (\tau_w - \tau_b) \cdot ds, \]  \hspace{1cm} (5)

can be used to reach a simple and important balance. The term on the left hand side corresponds to the mean transport cross the curve \( C \) which must be zero due to conservation of mass. The first term on the right hand side is also zero since it is the difference in surface elevation in the start and end points of the integration around a closed curve. Thus the surface wind stress and the bottom friction, or the surface and bottom layer Ekman transport, balance each other in an area enclosed by \( C \),

\[ \oint_C \tau^w \cdot ds = \oint_C \tau^b \cdot ds. \]  \hspace{1cm} (6)

Using a linear bottom friction,

\[ \tau^b = \mu v \]  \hspace{1cm} (7)

where \( \mu \) is a constant friction parameter, we can from the balance in equation (6) directly obtain the scale for the velocity component parallel to \( C \),

\[ v_p \sim \frac{\tau^w}{\mu}. \]  \hspace{1cm} (8)
The flow along contours of constant $\frac{f}{H}$ is consequently inversely proportional to the bottom friction parameter.

To scale the velocity component normal to $\mathcal{C}$ we take the curl of (3). The first term (divided by $f$) is then

$$\nabla \times (k \times \mathbf{V}) = \nabla \cdot \mathbf{V} = 0$$

according to the continuity equation (4). Note that $\nabla$ is still the horizontal gradient operator. Without loss of generality, $f$ can be taken as constant, in order that the second term can be rewritten according to

$$\nabla \times \left( -\frac{g}{f} H \nabla \eta \right) = -\frac{g}{f} \frac{\partial}{\partial x} \left( H \frac{\partial \eta}{\partial y} \right) + \frac{g}{f} \frac{\partial}{\partial y} \left( H \frac{\partial \eta}{\partial x} \right) = \mathbf{v}_g \cdot \nabla H$$

where $\mathbf{v}_g \equiv \frac{g}{f} k \times \nabla \eta$ is the depth-integrated geostrophic velocity. The result is thus

$$0 = \mathbf{v}_g \cdot \nabla H + \nabla \times \left( \frac{\tau^w - \tau^b}{f} \right).$$

Assuming small friction ($\mu \to 0$) enables a scaling of the velocity normal to contours of $H$,

$$\mathbf{v}_\perp \sim \frac{\tau^w}{f \Delta H}$$

where $\Delta H$ is the scale for the slope of the bathymetry.

The ratio of the velocity components now give

$$\frac{\mathbf{v}_\perp}{\mathbf{v}_\parallel} \sim \frac{\mu}{f \Delta H} \ll 1$$

for small friction ($\mu \to 0$) and steep bottom topography, i.e. the velocity is to the first order directed along lines of constant $\frac{f}{H}$.

The ratio $\frac{f}{H}$ can be recognized as the planetary part of the potential vorticity,

$$q = \frac{\zeta + f}{H},$$

where $\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$ is the relative vorticity. Flow along contours of $\frac{f}{H}$ is thus consistent with keeping the planetary part of the potential vorticity constant. In the Arctic Ocean and Nordic Seas, such contours are dominated by topography and thus closed $\frac{f}{H}$-contours exist and remotely control a wind-induced large-scale circulation.
2.2 An analogy with isobars

Without loss of generality, we again set $f$ constant. Assuming a geostrophic relation, flow along isobaths then states that the pressure field must be a function of $H$ alone, i.e. $\eta = \eta(H)$. Considering a closed curve $C$ on which $H$ is constant and setting a small line segment along $C$ to $ds$ and a small line segment normal to $C$ to $dn$ makes it possible to rewrite the geostrophic relation $\mathbf{v} \equiv \frac{\partial}{\partial f} \mathbf{\kappa} \times \nabla \eta$ as

$$\mathbf{v} = \frac{g}{f} \frac{d\eta}{dH} \frac{dH}{dn} \text{s.}.$$  \hspace{1cm} (15)

If this is put as the velocity in the linear friction $\tau^b = \mu \mathbf{v}$ in the balance in equation (6), we get the following

$$\oint_C \tau^w \cdot ds = \frac{d\eta}{dH} \oint_C \frac{\mu g}{f^2} \frac{dH}{dn} ds.$$  \hspace{1cm} (16)

Since the velocity was found to be proportional to $\tau^w$ (see equations (8)), we can from this deduce that when the bathymetry is steep ($\frac{dH}{dn}$ is large) the speed of the flow is high, which can be compared to how strong gradients in isobars are associated with high speeds.

3 An equatorial basin

Imagine a barotropic rectangular basin with low friction and a bowl-like shaped bathymetry with continental slopes on all four boundaries, in which the dynamics are governed by the closed contours of $\frac{f}{H}$. Now let the Equator cross the middle of the basin and assume an equatorial beta-plane $f = \beta y$. In such a basin the contours of $\frac{f}{H}$ no longer close on themselves, but instead converge and meet at the intersection of the Equator and the western and eastern coasts, where both the Coriolis parameter and the depth go to zero. See figure 3.

Assume also a localized wind stress with positive curl north of the Equator. The circulation induced by this will then, as motivated in section 2, follow lines of constant $\frac{f}{H}$ westward to the slope and turn south-west onwards toward the equator.
3.1 A frictional boundary layer

Salmon (1992) described the flow on a western continental slope with a southern and northern boundary present. In that case, the southern boundary lay north of the Equator. In this section, a course of action similar to that of Salmon will be taken, with the difference that the southern coast lies far south of the Equator; dynamics at the equator will thus be subject to consideration.
Inspired by Salmon (1992), the curl of (3) can be rewritten as
\[ \mathbf{V} \cdot \nabla \frac{f}{H} = \nabla \times \left( \frac{\tau^w}{H} - \frac{\mu \mathbf{V}}{H^2} \right) \] (17)
where the linear bottom friction has been written as \( \tau^b = \mu \mathbf{v} \). If the transport stream function \( \psi \), defined as \( \mathbf{V} \equiv \mathbf{k} \times \nabla \psi \), is introduced, the equation can be written
\[ \mathbf{V} \cdot \nabla \frac{f}{H} = -\nabla \cdot \left( \frac{\mu}{H^2} \nabla \psi \right) + \nabla \times \left( \frac{\tau^w}{H} \right). \] (18)

Let us qualitatively discuss the importance of the terms on the right hand side of (18) on the western continental slope. On the western slope the wind stress can be neglected; either because it doesn’t exist there (assume for instance a localized wind in the interior of the north part of the basin) or because we can assume that the fluid velocities there make it negligible. On the slope away from the equator, we can, since the friction is small, assume that the right hand side is small, i.e.
\[ \mathbf{V} \cdot \nabla \frac{f}{H} \approx 0. \] (19)

The transport, and thus lines of constant transport stream function, are therefore along contours of \( \frac{f}{H} \) on the slope. In the vicinity of the Equator, though the friction parameter \( \mu \) is small, the gradient of the stream function grows as the flow follows the converging contours of \( \frac{f}{H} \). The friction term also grows as a direct effect of the decreasing depth. In a small area close to the Equator, the term on the left hand side of (18) is therefore balanced by the friction term. This implies transport cross the contours.

To analyse how the Equator affects the circulation in the basin, we linger in this thin boundary layer where the western boundary intersects the Equator and where friction becomes important. Assume that the depth \( H \) on the western slope can be written as \( H = sx \) and the Coriolis parameter \( f \) as \( f = \beta y \). Further assume a polar coordinate system \((x = r \cos(\theta), y = r \sin(\theta))\) with \( r = 0 \) where the Equator intersects the western boundary. Equation (18) can then be written
\[ \frac{\beta}{s \cos^2(\theta)} \frac{1}{r} \frac{\partial \psi}{\partial r} = -\frac{\mu}{s^2 \cos^2(\theta)} \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial \psi}{\partial r} \right) - \frac{\mu}{s^2 r^3} \frac{\partial}{\partial \theta} \left( \frac{1}{\cos^2(\theta)} \frac{\partial \psi}{\partial \theta} \right). \] (20)
First let us, without any strict motivation, neglect the last term involving derivatives with respect to $\theta$. We will come back to the validity of this later.

To the second order differential equation with a dependence on $r$ alone it can easily be verified that a solution is

$$\psi_B = \psi_0 e^{-\frac{\beta s}{2\mu} r^2}$$

(21)

where $B$ indicates that it is only valid in the boundary layer and $\psi_0$ is a constant amplitude. This is a solution decaying fast as we move away from the boundary layer.

To demonstrate that this is the only solution, given some boundary conditions, let us make an intuitive assumption that the amplitude $\psi_0$ is a function of $\theta$ and that it is explicitly determined by the wind stress curl in the interior. I.e. we interpret it as a wind-induced flow that impinges the slope for some $\theta$ and then travel along lines of constant $f_H$, which on the slope are also lines of constant $\theta$. The total stream function, $\psi$, would then be a composition of the interior and boundary layer stream functions. Let’s say

$$\psi(r, \theta) = \psi_0(\theta)(1 - \psi_B(r)).$$

(22)

Now, a boundary condition is $\psi(0, \theta) = 0$, which leaves the condition on the boundary layer stream function that $\psi_B(0) = 1$. It can be shown that (21) is the only solution satisfying this condition.

The coefficient in the exponent of (21) can be taken as the inverse of the square of a length scale, $L$, for the boundary layer, i.e.

$$L^2 = \frac{\mu}{\beta s}.$$  

(23)

The size of the boundary layer would thus decrease for a smaller friction parameter and a steeper slope.

So far the friction has been parametrized so that the equation of motion would look like

$$\frac{\partial \mathbf{v}}{\partial t} + ... = -\frac{\mu \mathbf{v}}{H(x,y)} + ...$$

(24)

An alternative to the parametrization in (24), is to ignore the varying depth and define a time scale $t_{fric} \equiv \frac{H_{max}}{\mu}$, where $H_{max}$ is the interior maximum depth;

$$\frac{\partial \mathbf{v}}{\partial t} + ... = -\frac{\mathbf{v}}{t_{fric}} + ...$$

(25)
3.2 On the more general solution

The friction thus has a constant value throughout the basin. The depth in the friction term in equation (18) would then lose the square. An analogous reasoning to above leads to a solution

$$\psi_B = \psi_0 e^{-\frac{\beta H_{max}}{\mu \cos(\theta)} r},$$

where the exponent now is independent of the bottom topography and gives a length scale

$$L = \frac{\cos(\theta)}{\beta \ t_{fric}}.$$  \hspace{1cm} (27)

To compare the two different parametrizations (24) and (25) and their respective boundary layer length scales $L_{depth}$ and $L_{const}$, we write

$$\frac{L_{const}}{L_{depth}} = \frac{\mu \cos(\theta)}{\beta H_{max} \sqrt{\frac{\mu}{\beta s}}} = \frac{sL_{depth} \cos(\theta)}{H_{max}} \leq \frac{L_{depth}}{W} < 1 \hspace{1cm} (28)$$

if the boundary layer length scale is smaller than the width $W = \frac{H_{max}}{s}$ of the slope. Thus, the friction parametrization that is independent of depth results in a thinner boundary layer. This can also be realized by considering that a friction that is comparable in the interior, would for the depth dependent parametrization result in a higher effective friction closer to the coast where the depth is shallower.

3.2 On the more general solution

To come back to the neglected terms with derivatives with respect to $\theta$, we now scale (20) with the horizontal length scale of the boundary layer $L^2 = \frac{\mu}{\beta s}$ so that we can write it in non-dimensional form as

$$\frac{1}{r \cos^2(\theta)} \frac{\partial \psi}{\partial r} = -\frac{1}{r \cos^2(\theta)} \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial \psi}{\partial r} \right) - \frac{1}{r^4} \frac{\partial}{\partial \theta} \left( \frac{1}{\cos^2(\theta)} \frac{\partial \psi}{\partial \theta} \right)$$  \hspace{1cm} (29)

where $r$ is now a non dimensional parameter in the length $rL$. With respect to $r$, the terms on the right are of the same order of magnitude, i.e. we have no reason to assume that the derivatives with respect to $\theta$ can be neglected.
If we separate the variables in equation (29), with \( \psi(r, \theta) = F(r)G(\theta) \), the resulting differential equations are

\[
\begin{align*}
    r^2 F'' + (r^3 - r) F' - \lambda^2 F &= 0 \quad (30) \\
    G'' + 2 \tan \theta G' + \lambda^2 G &= 0 \quad (31)
\end{align*}
\]

where \( \lambda \) is a constant. For the special case of \( \lambda = 0 \), equation (30) corresponds to the same equation as we had when we omitted the terms involving derivatives with respect to \( \theta \) in (29). However, nothing has been stated to support this neglect or this value of \( \lambda \). The simple solution (21) found above might therefore not be dominating in the complete differential equation, but it is not unlikely that the boundary layer thickness discussed still has some reliability.

It is reasonable to think that solutions to (30) and (31), together with a solution for the interior, can fully describe the circulation. However, an analytical description of this will not be within the scope of this report, but some aspects of the radially dependent solution will be briefly stated. The solution to equation (30) is (Kamke, 1951)

\[
F(r) = r e^{-\frac{r^2}{4}} Z_\nu(r^2/4) \quad (32)
\]

where \( Z_\nu \) are modified Bessel functions with order \( \nu \) where \( 4\nu^2 = \lambda^2 + 1 \). The solution given by equation (21), that corresponds to \( \lambda = 0 \), will here be given by \( \nu = \pm \frac{1}{2} \). Unlike the ordinary Bessel functions, which are oscillating functions, the modified Bessel functions are exponentially growing respectively decaying.

4 Numerical experiments

To make a continued analysis of the wind-induced circulation in the equatorial basin, simulations were run with a depth-integrated nonlinear shallow water model.

4.1 The model

The model used is a barotropic shallow water model by Döös et. al (2004). It is built on a C-grid with spherical coordinates. For all usage in this context, the model is set to nonlinear with an energy conserving scheme. The resolution is 0.25 degrees and the time step is 40 seconds.
The model has a super ellipse shaped bathymetry with a flat bottom and linear continental slopes. Here with a maximum height of 1000 m.

The domain is placed over the equator from 30°S to 30°N and it stretches over 60 longitudes. The bottom topography is built with linear continental slopes on all four boundaries, where the contours of $H$ are shaped like a super ellipse, e.g.,

$$x^4 + y^4 = 1.$$  \hspace{1cm} (33)

The slope has a certain width that corresponds to about one sixth of the total domain side length and surrounds an interior with flat bottom where the maximum depth is 1000 m. See figure 4. In the corners of the quadratic domain, outside of the outermost super ellipse shaped depth contour, the depth is set to a constant shallow depth of 10 m.
4.1 The model

Figure 5: A description of the zonal wind stress as it is implemented in the model. In the left figure is the zonal wind stress, $\tau_x$, and in the right figure is the curl of the wind stress, $-\frac{\partial \tau_x}{\partial y}$, which is negative in the upper half of the patch and positive in the lower.

A zonal wind stress, $\tau_x^w$, is placed in a patch north of the Equator. To avoid sharp discontinuities in the curl of the wind stress, the shape is chosen to be parabolic in the y-direction. The stress can be described as

$$\tau_x^w(x, y) = \begin{cases} 
\tau_0 \left( -1 + \frac{(y-y_0)^2}{l^2} \right) & \text{for } -l < y - y_0 < l, \ x_1 < x < x_2 \\
0 & \text{elsewhere}
\end{cases}$$

(34)

where $2l$ is the meridional width of the patch, $y_0$ is situated in order that the area in which the wind is non-zero lies entirely north of the equator and $x_1$ and $x_2$ are chosen arbitrarily. See figure 5. The wind is thus easterly and the parabolic shape of the stress in the y-direction makes the curl of the wind stress, $-\frac{\partial \tau_x}{\partial y}$, positive in the southern half of the patch and negative in the northern half. The resulting Sverdrup transport (see equation (1)) is thus directed toward the middle of the patch.

A linear friction according to (25) is used as standard. The time scale for the bottom friction, $t_{\text{fric}}$, is set to 10 days. If the time scale is defined as $t_{\text{fric}} \equiv \frac{H_{\text{max}}}{\mu}$ this would give that $\mu$ is of the order of $10^{-3}$ m/s. For a continental slope of maximum depth of 1000 m and $f = 7 \cdot 10^{-5}$, the ratio (see equation (13))

$$\frac{\mu}{f \Delta H} \sim 10^{-2},$$

(35)

i.e. the experiments are constructed so that the dominating velocity component is along lines of constant $\frac{f}{H}$. 15
To analyse the steady state circulation simulations are run for 50 model days, i.e. five times the friction time scale. Steady state will henceforth refer to the last time step of a simulation run for 50 days.

4.2 Results and discussion

In figure 6 the steady state height of the free surface is presented. The easterly wind, constructed according to (34), is situated approximately where the maximum and minimum in surface height are encountered. The first thing to notice is that there is no signal south of the equator. Before this is discussed further, the basic circulation will be described.

The velocity field is shown in figure 7. There is no signal south of the equator here either, except close to the intersection of the equator and the western boundary. In an area between about 5°E and 15°E and 10°N and 20°N the wind stress is applied. In this area, the wind forcing induces a weak north-south Sverdrup drift toward the center of the patch. The induced flow outside of the patch is a westward flow originating at the western edge of the patch. As the flow impinges on the slope and turns in a south-west direction, it becomes evident that it follows contours of $\frac{f}{\beta}$, cf. figure 3. Along the slope the speed increases until a thin jet forms near the equator, giving transport cross the contours. This seems to resemble a frictional boundary layer as discussed in section 3.1 (cf. figure 1). A more thorough discussion on this can be found in section 4.2.2 below. In the boundary layer, the flow turns around and closes the circulation by returning to the Sverdrup drift from the south, again along contours of $\frac{f}{\beta}$. A similar closed circulation is also formed north of this, due to the north-south symmetry of the wind, but this is only considered a by-product of the wind field and will not be the focus of this study.

In the interior, where the bottom is flat, and on the slope away from the equator, comparison between figure 6 and 7 implies geostrophic motion. Near the equator, however, the pressure field gives no hint of the boundary layer jet that is visible in the velocity field, indicating non-geostrophic flow. The ratio between the time scales for friction and coriolis can be written $f_{\text{max}}t_{\text{fric}}$ and gives a measure of the importance of the two terms in the governing equations. Away from the equator, $f_{\text{max}} = 7 \cdot 10^{-5} \text{ s}^{-1}$ and $t_{\text{fric}} = 10 \text{ days}$ gives $f_{\text{max}}t_{\text{fric}} > 1$, indicating geostrophy. Closer to the equator, the coriolis parameter is rather of the order of $10^{-7} \text{ s}^{-1}$, resulting in $f_{\text{max}}t_{\text{fric}} < 1$ which indicates that friction becomes important.

The transport stream function, defined as $\mathbf{V} \equiv \mathbf{k} \times \nabla \psi$, is shown in figure 8. The stream lines in general follow lines of constant $\frac{f}{\beta}$, though there are
clear deviations. One area of cross contour transport is already mentioned and is in the proximity of the western boundary, where the two jets are situated in figure 7. The jets, and the boundary layer, are however not as clear in the transport stream function as in figure 7. Though the velocities have their maximum values in the vicinity of the equator, the shallow depth near the coast makes the depth-integrated transport less pronounced. The cross contour transport is instead evident across the entire slope, and is, as implied earlier, due to friction. This indicates that equation (19) is not really satisfied. It can be speculated that a clearer boundary layer might require a lower friction than what is implemented here, but to search for the best friction parametrization or strength of friction is not within the scope of this project. In the interior, with the strongest indication near the discontinuity in the wind field, there is also a cross contour component of the transport.

4.2.1 A southern hemisphere response?

Except in the narrow area of the boundary layer that stretches south of the equator, all of the south hemisphere is found to be stagnant in the steady state. As is the case for the areas east of the wind stress patch, i.e upstream in the sense of the propagation of long Rossby waves. The circulation described above is similar to the Stommel model (cf. figure 1), but compressed southward toward the equator; it follows contours to a frictional boundary layer with cross contour flow and then closes the circulation by returning along another contour. This can be interpreted as a similar balance as in equation (6) but with piecewise constant $f_H$.

Earlier it was speculated that the flow might travel from the intersection of the equator and the western boundary along a contour in the southern hemisphere. The simulated flow seem to rather follow a contour on the same hemisphere back from the equatorial region, than a connecting contour south of the equator. Why this is, might be made more clear by looking at the vorticity of the flow. The unforced equations governing a shallow water system conserves potential vorticity (Vallis, 2006). With small friction and outside of the wind forced area (i.e. where the flow follows contours), an approximation is that the relative vorticity should not change, since this would alter the potential vorticity. The ratio $\frac{\zeta}{f_H \frac{H_{max}}{f_{max}}}$, i.e. the ratio between the contributions to the potential vorticity (see equation (14)) from the relative vorticity and the maximum planetary vorticity, is shown in figure 9.
4.2 Results and discussion

Figure 6: Height of free surface [m] at steady state. The axis are in degrees longitude resp. latitude.

The flow has little or no relative vorticity where it follows lines of constant $f$ in the interior away from the wind. The presence of relative vorticity is highly correlated to where there is also friction or wind forcing acting, i.e. where there is cross contour transport. Where the wind stress acts, the relative vorticity seem to be, in some sense, proportional to the curl of the wind stress, i.e. positive in the southern half of the patch and negative in the northern (see (34)).
4.2 Results and discussion

NUMERICAL EXPERIMENTS

Figure 7: Flow velocity [m/s] at steady state. The axis are in degrees longitude resp. latitude. Note that the equator lies in the lower half of the figure.

Flow with small relative vorticity thus impinges on the slope. Where the southward jet crosses the contours, the relative vorticity has increased along the slope to a maximum value. For the flow to pass over the equator and continue along a contour in the southern hemisphere (if all energy was not dissipated in the frictional boundary layer) the planetary vorticity would have to become negative. This would in turn imply that the relative vorticity of the flow would have to increase enough to compensate. From figure 9 it is clear that it does increase, but it seems like it is only enough for the circulation to just advance over the equator but then turn and follow a contour with a lower value of $f/\sin \phi$ in the return flow.
4.2 Results and discussion

The transport stream function \([m^3/s]\) at steady state. Positive values are drawn solid and negative values are dashed. The axis are in degrees longitude resp. latitude. Note that the equator lies in the lower half of the figure.

The potential vorticity contours are largely dominated by \(\frac{f}{H}\). The maximum contribution from relative vorticity is encountered in the boundary layer, where it is a few tenths of the planetary contribution.

If the relative vorticity were to increase enough, by some means that will not be discussed here, the circulation south of the equator would be strongly anticyclonic and the present author has difficulties imagining a sensible steady circulation with these properties.
4.2 Results and discussion

4.2.2 Horizontal length scale of the boundary layer

In section 3.1, two different parametrizations of the friction were discussed. To make a numerical analysis, a simulation with the friction parametrization in (24), that takes the bathymetry into account, was run to be compared to the above discussed simulation. The friction parameter, \( \mu \), was then set to 0.0004 ms\(^{-1}\). This corresponds to a time scale \( t_{fric} \equiv \frac{H_{\text{max}}}{\mu} \) of about 30 days, i.e. three times longer than in the simulation described above. However, it should be noted that the effective friction near the boundaries will be considerably larger than in the interior because of the shallower depths.

The stream function from this simulation is shown in figure 10. The tendency to follow the contours seems more pronounced in the interior compared to figure 8. Since the friction in this case is lower, this, again, suggests that cross contour flow is a frictional effect.

The theoretical analysis in section 3 resulted in expressions for the horizontal length scales of the boundary layer. For the depth dependent friction parametrization, the expression (23) gives

\[
L = \sqrt{\frac{\mu}{\beta s}} \sim 200 \text{ km or } 2^\circ
\]  

(36)

for \( \mu = 0.0004 \text{ ms}^{-1}, \beta = 10^{-11} \text{s}^{-1} \text{m}^{-1} \) and the slope \( s = \frac{H_{\text{max}}}{W} = \frac{1000}{10^9} \). The alternative parametrization length scale (27) gives

\[
L = \frac{\cos(\theta)}{\beta t_{fric}} \sim 80 \text{ km or } < 1^\circ
\]  

(37)

for \( \cos(\theta) \sim \cos(\pi/4), \beta = 10^{-11} \text{s}^{-1} \text{m}^{-1} \) and \( t_{fric} = 10 \text{ days} \). This is, as the reasoning in (28) also states, thinner than the former. Although the friction according to \( t_{fric} \) is smaller for the depth dependent parametrization, the size of the boundary layer is still larger.
Figure 9: $\frac{\zeta}{f_{\text{max}}}$, i.e. the contribution to the potential vorticity from the relative vorticity, nondimensionalized by the maximum contribution from the planetary vorticity, in steady state. The axis are in degrees longitude resp. latitude. The upper panel shows the wind-driven area and the lower panel the boundary layer.
The above is the results from the analytical predictions with parameter values from the simulations. To compare it to the actual simulated numerical length scales, cross sections through the boundary layer for the transport stream functions from the two simulations are shown in figure 11. The cross section is taken from the intersection of the equator and the western boundary, in a north-east direction along $x = \frac{3y}{2}$ and a distance of about 6° longitude (to be compared to the width of the slope which is about 10°). This corresponds to $\psi$ as a function of $r$, where $r$ is the radial distance from the intersection, and approximates a line of constant $\frac{f}{H}$ to some extent (see the red lines in figure 11). In the figure are also the analytical transport stream functions (dashed) according to

\[
\psi = 1 - e^{-\frac{r^2}{4\pi^2}} \quad (38)
\]
\[
\psi = 1 - e^{-r/L} \quad (39)
\]

(cf. (22)), which have been nondimensionalized by their respective boundary layer length scales $L$. The stream functions from the simulations have been nondimensionalized by their maximum value on the slope along the cross section.

The numerically simulated stream functions show a boundary layer behaviour, with a steep increase with increasing $r$ close to the boundary, that levels out as we exit the boundary layer. This corresponds to strong cross contour flow in the boundary layer, if $r$ is thought of as an approximation to contours of $\frac{f}{H}$. The behaviour of the stream functions do, however, differ in shape between the parametrizations. Notable is also their resemblance in shape to their respective analytical solutions, where the depth dependent parametrization shows a more Gaussian structure, just as the analytical solutions suggested. This indicates that the assumption of negligible angular dependence (see section 3.1) might be tenable. Figure 11 suggests that the simulated boundary layers are wider than the analytical length scales, since their local maximums lies further from the equator. However, it should be noted that the cross section of the simulated stream function is not necessarily taken along a true contour of $\frac{f}{H}$, nor is it an isolated boundary solution. It has also been noted that cross contour flow is evident over the entire slope. Consequently, there are reasons for why the analytical solutions might deviate from the numerical solutions in figure 11 and it is possible that the analytic solution better resembles the numerical than the figure suggests.
Figure 10: The transport stream function $[m^3/s]$ at steady state from a simulation with a friction parametrization that is depth dependent, see (24). Positive values are drawn solid and negative values are dashed. The axis are in degrees longitude resp. latitude. Note that the equator lies in the lower half of the figure.

The cross sections are taken an equal distance out from the equator and consequently the length scales in the two figures can be compared to each other. Such a comparison reveals the already established result that the analytically predicted length scale for the depth dependent parametrization is larger. The same conclusion can, however, not as easily be drawn for the numerical length scales. But, as mentioned above, there are reasons not to expect a perfect match between theoretical results and the model results in figure 11.
4.2 Results and discussion

4.2.3 Preliminary results

Some additional aspects that were considered, but, due to limited time, were not fully investigated during the project will be discussed briefly in this section.

Since the analytical analysis was linear but the numerical model nonlinear, differences between results could be due to nonlinearities. Experiments with the purpose of examining the influence of nonlinearities were therefore conducted. A less advanced linear shallow water model was run, but no significant differences in the steady circulation was found. This could imply a small Rossby number. The amplitude of the wind forcing in the nonlinear model was decreased and increased respectively, as well, in order to change the Rossby number, but with no indication of significant changes in the steady circulation. However, the experiments were not as thorough, nor did the time suffice to make them so, as to be able to draw any certain conclusions from.

Experiments were also conducted in an attempt to trigger an upstream (with respect to the westward propagation of Rossby waves) or southern hemisphere response by applying wind on the equator. For example, wind according to (34) but with opposite sign, i.e. westerly, was applied with the center of the patch on the equator. The Sverdrup transport, proportional to the curl of the wind stress, is then northward north of and southward south of the equator, i.e. the transport is outwards from the equator. A strong eastward jet along the equator was induced, which can be compared to the Fofonoff model (e.g. Vallis, 2006), but still no upstream signal was produced.
Figure 11: Cross section of the transport stream function at steady state (solid line) and the analytical transport stream function according to 22 (dashed), both non-dimensionalized. The x-axis is $r/L$, where $r$ is the radial distance from the equator along the cross section and $L$ is the analytical length scale of the resp. boundary layers. The cross section, illustrated by the red line in the figures to the right, is taken from the intersection of the equator and the western boundary and in a north-east direction as $x = \frac{3y}{2}$. Upper panel: Friction parametrization without depth dependence, see (25). Lower panel: Friction parametrization with depth dependence, see (24).
5 Main conclusions

The numerical analysis show that a localized wind north of the equator does not induce any steady circulation south of the equator. The equator seem to play a similar role as does the vertical western boundary in the Stommel model (cf. figure 1); a thin frictional boundary layer forms where the equator intersects the western boundary, allowing the flow to cross the contours and enable a return flow, again along contours north of the equator.

In terms of potential vorticity, it is dominated by lines of constant $f_H$. The relative vorticity contributes to the potential vorticity where horizontal stresses are important, i.e. where the wind is applied and in the frictional boundary layer, but the contribution is not in the same order of magnitude as the contribution from the planetary vorticity. Thus, in no experiment in this study, the relative vorticity has increased enough for the flow to cross the equator and follow contours of $f_H$ in the southern hemisphere. It could be speculated that a more advanced model or a different friction parametrization could result in different conclusions, but the present author is sceptical to a southern hemisphere response.

Analytical solutions for the frictional boundary layer were examined. Two different linear friction parametrizations give the solutions

$$\psi = \psi_0 \left(1 - e^{-\frac{bs}{2\mu}r^2}\right)$$

$$\psi = \psi_0 \left(1 - e^{-\frac{b_{fric}}{\cos(\theta)}r}\right)$$

for one friction parametrization dependent and one independent of the varying depth, respectively. Here, $\psi$ is the transport stream function, $r$ is the radial and $\theta$ the angular coordinate, $\psi_0$ is a constant amplitude, $\beta$ is the gradient of the Coriolis parameter, $s$ is the linear slope of the bathymetry, $\mu$ is the constant friction parameter and $t_{fric} = \frac{H_{max}}{\mu}$ is the friction time scale. The exponential behaviour with $r^2$ and $r$, respectively, was also recognized in the correspondent numerical solutions in the boundary layer, indicating reliability of the presented analytical solutions.

The exponents of these solutions give horizontal length scales of the boundary layer that are also dependent respectively independent of the slope. The expressions are

$$L^2 = \frac{\mu}{\beta s}$$

$$L = \frac{\cos(\theta)}{\beta t_{fric}}.$$
Acknowledgements

To Johan Nilsson, for the idea of the subject, for his support and guidance through the project and for his generous additional education in oceanography. To Kristofer Döös, who dusted off an old model for me, and to Laurent Brodeau who helped with the dusting.
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